

Multiple target tracking based on sets of trajectories

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Abstract—This paper proposes the set of target trajectories as the state variable for multiple target tracking. The main objective of multiple target tracking is to estimate an unknown number of target trajectories given a sequence of measurements. This quantity of interest is perfectly represented as a set of trajectories without the need of arbitrary parameters such as labels or ordering. We use finite-set statistics to pose this problem in the Bayesian framework and formulate a state space model where the random state is a random finite set that contains trajectories. All information of interest is thus contained in the multitrajectory filtering probability density function (PDF), which represents the multitrajectory PDF of the set of trajectories given the measurements. For the standard measurement and dynamic models, we describe a family of PDFs that is conjugate in the sense that the multitrajectory filtering PDF remains within that family during both the prediction and update steps.

Index Terms—Bayesian estimation, multiple target tracking, random finite sets.

I. INTRODUCTION

Multiple target tracking (MTT) has an extensive range of applications, for example, in surveillance [1], robotics [2] or computer vision [3]. Its main goal is to estimate the states at different time steps of an unknown and variable number of targets building a sequence of estimates for each target. Such sequences are called trajectories or tracks. In this paper, we are primarily interested in how the MTT problem is formulated, that is, the selection of the state variable that best represents the quantity we want to estimate. We proceed to review state representations used in the literature, along with their pros and cons, before stating our contributions and their practical implications.

Classic approaches to MTT, such as multiple hypothesis tracking (MHT) [1], [4], [5] and joint probabilistic data association (JPDA) [6], [7], represent the state at a given time step as a vector. Currently, it is widely accepted that using a set representation in multi-object systems has some advantages over vector representation, such as the possibility to define metrics for algorithm evaluation [8] and avoiding an arbitrary ordering of the objects. The random finite set (RFS) framework and finite-set statistics (FISST), developed by Mahler, constitute an appealing and rigorous way of dealing with such multiple object systems from a Bayesian point

of view [9], [10]. In the classic RFS algorithms, the state at a certain time step is the set of targets at this time step and the main focus has been on the filtering problem. That is, we calculate/approximate the multitarget probability density function (PDF) of the current set of targets given the measurements [10], which we refer to as multitarget filtering PDF. Based on the multitarget filtering PDF, we can estimate the set of targets at the current time but it is not obvious how to build trajectories in a sound manner.

The most popular approach to try to build trajectories from first principles consists of adding a label to each single target state so that the state at a certain time step becomes the set of labelled targets [11]–[16]. Labels are unique, unobservable and static quantities which identify a target over its life time. Rather than following an optimal Bayesian approach by obtaining trajectory estimates directly from a multiobject PDF that represents the available knowledge about the trajectories, the standard approach is to estimate the set of labelled targets at each time step using the labelled multitarget filtering PDF. Track estimates are then formed simply by linking target state estimates with the same label.

While the previous two-step procedure with sets of labelled targets yields acceptable results in many situations, there can be ambiguity in track formation, which results in track switching or track coalescence [17]. For example, track ambiguity arises if several targets move in close proximity for a sufficiently long time and then separate [18]. This situation is important because tracking algorithms should be able to handle the problem of formation-flying [5], [19]. Track ambiguity also happens if new born targets are modelled as an independent and identically distributed (IID) cluster RFS or Poisson RFS [10], [20], [21]. This problem is caused by the fact that any labelling of the estimates of targets born at the same time is equally possible according to the multitarget filtering PDF [18].

A possible solution to track ambiguity is to extend the state representation to the sequence of set of labelled targets at all time steps. In this case, we can estimate trajectories based on the PDF of this state given all measurements, as pointed out in [14, Sec. II.B] or used implicitly in [22]. While the sequence of set of labelled targets contains all relevant information to build trajectories and can therefore lead to optimal trajectory estimation, it contains arbitrary variables: the labels. The inclusion of arbitrary labels in trajectory states is unnecessary and prevents the definition of metrics with physical interpretation, as explained in Section II-B.

In this paper, we make the following contributions. First of all, our main contribution is that we propose the set of trajectories as the state variable for MTT. In MTT, we are primarily interested in estimating the number of trajectories,

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start times, lengths and sequence of target states for each trajectory as these variables contain complete information about the movement of targets. A set of trajectories provides a minimal, unambiguous representation of the previous quantities of interest, without arbitrary variables such as labels. Second, we explain how to use Mahler's RFS framework [10], which is not restricted to RFS of targets, for RFS of trajectories to perform Bayesian inference. Third, we present the filtering equations and a conjugate family of PDFs, in the spirit of [13], for computing the multitrajectory filtering PDF. Fourth, we clarify the meaning of the labels by showing that they are hidden variables that appear in the multitrajectory filtering PDF with the standard measurement model. Fifth, we establish the relation between the multitrajectory filtering PDF and the multitarget filtering PDF. Sixth, we demonstrate that the closed-form solution to the filtering problem using labelled RFS [13] presents track switching with an IID cluster birth process while it does not happen for sets of trajectories.

Apart from these theoretical contributions, the adoption of set of trajectories as state variable enables the development of metrics with physical interpretation [23] and new algorithms, such as the trajectory probability hypothesis density (PHD) filter [24]. The trajectory PHD filter, is an extension of the PHD filter [10] and builds trajectories from first principles based on the propagation of a Poisson process on the space of sets of trajectories through the filtering recursion. Both the trajectory PHD filter and PHD filter are intrinsically unlabelled. Preliminary results on MTT with sets of trajectories were provided in [25].

The rest of the paper is organised as follows. In Section II, we define a set of trajectories and motivate its use as the state variable. In Section III, we provide the recursive equations to calculate the multitrajectory filtering PDF. We analyse the relations among the proposed approach, labelled approaches, the usual RFS tracking framework based on unlabelled targets and classical MHT in Section IV. An illustrative example is provided in Section V and concluding remarks are given in Section VI.

II. SETS OF TRAJECTORIES

In this section, we introduce the variables, motivate why we propose the set of trajectories as a state variable and indicate how to use FISST for sets of trajectories.

A. State variables and notation

A single target state $x \in D$, where $D = \mathbb{R}^{n_x}$, contains the information of interest about the target, e.g., its position and velocity. A set \mathbf{x} of single target states belongs to $\mathcal{F}(D)$ where $\mathcal{F}(D)$ denotes the set of all finite subsets of D . We are interested in estimating all target trajectories, where a trajectory consists of a sequence of target states that can start at any time step and end at any time after it starts. Mathematically, we represent a trajectory as a variable $X \in T$, where $T = \mathbb{N} \times \uplus_{i=1}^{\infty} D^i$, D^i represents i Cartesian products of D and \uplus stands for disjoint union, which is used in this paper to highlight that it is the union of disjoint sets, as in [26]. We write $X = (t, x^{1:i})$ where t is the initial time of the trajectory,

i is its length and $x^{1:i} = (x^1, \dots, x^i)$ denotes a sequence of length i that contains the target states at consecutive time steps of the trajectory. Similarly to the set \mathbf{x} of targets, we denote a set of trajectories as $\mathbf{X} \in \mathcal{F}(T)$.

Given a single target trajectory $X = (t, x^{1:i})$, the set $\tau^k(X)$ of the target state at time k is

$$\tau^k(X) = \begin{cases} \{x^{k+1-t}\} & t \leq k \leq t+i-1 \\ \emptyset & \text{elsewhere.} \end{cases}$$

As the trajectory exists from time step t to $t+i-1$, the set is empty if k is outside this interval. We employ the following terminology:

- A trajectory X is present at time step k if and only if $|\tau^k(X)| = 1$.
- A surviving trajectory at time k is a trajectory that is present at times k and $k-1$.

Given a set \mathbf{X} of trajectories, the set $\tau^k(\mathbf{X})$ of target states at time k is

$$\tau^k(\mathbf{X}) = \bigcup_{X \in \mathbf{X}} \tau^k(X). \quad (1)$$

In RFS modelling, two or more targets at a given time cannot have an identical state [10, Sec 2.3]. The corresponding assumption for sets of trajectories is that any two trajectories $X, Y \in \mathbf{X}$ must satisfy that $\tau^k(X) \cap \tau^k(Y) = \emptyset$ for all $k \in \mathbb{N}$. Note that this assumption ensures that the cardinality $|\tau^k(\mathbf{X})|$ of $\tau^k(\mathbf{X})$ represents the number of trajectories present at time k .

Example 1. An example of a set of trajectories with one-dimensional target states is $\mathbf{X} = \{X_1, X_2, X_3\}$ with $X_1 = (1, (1, 1.5, 2))$, $X_2 = (1, (0.5, 0.625, 0.75, 0.875, 1))$ and $X_3 = (2, (2.4, 2.6, 2.8, 3))$, which is illustrated in Figure 1(top). There is one trajectory that starts at time 1 with length 3 and states $(1, 1.5, 2)$, another that starts at time 1 with length 5 and states $(0.5, 0.625, 0.75, 0.875, 1)$ and a third one that starts at time 2 with length 4 and states $(2.4, 2.6, 2.8, 3)$. We also have that, e.g., $\tau^1(\mathbf{X}) = \{1, 0.5\}$ and $\tau^5(\mathbf{X}) = \{3, 1\}$.

The information contained in \mathbf{X} is also found in a sequence of sets of labelled targets [14], as illustrated in Figure 1(bottom). For example, squares, crosses and circles can represent the target states with labels l_1, l_2 and l_3 , respectively. However, labels are arbitrary, as they do not represent any physical, meaningful quantity so we can make any other association, for example, squares- l_2 , crosses- l_1 and circles- l_3 to represent the same physical reality. \square

B. Motivation for sets of trajectories

In this section, we motivate the use of sets of trajectories as state variable in MTT.

In vector based state space models, it is often important to consider the PDF over the states at all time steps, i.e., it is not sufficient to merely find all the marginal PDFs of the states at all times [27], [28]. The same is true in MTT, where such a multitrajectory PDF would enable us to answer all trajectory related questions [27], for example, what is the probability

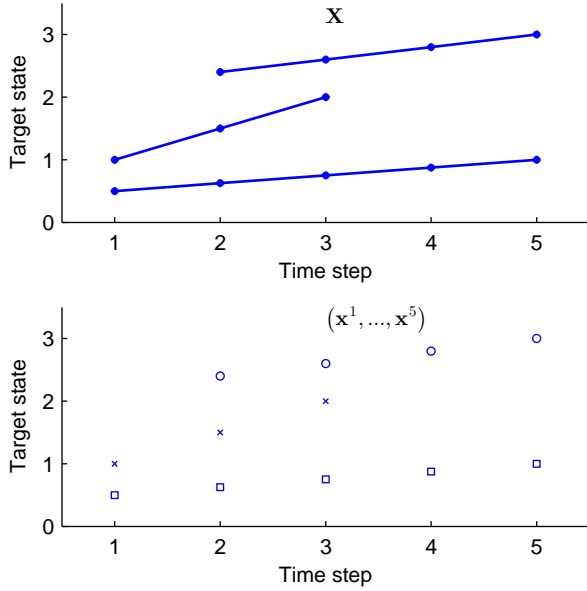


Figure 1: Illustration of set of trajectories of Example 1 (top) and an equivalent sequence of sets of labelled targets where squares, crosses and circles represent three different labels (bottom).

that a target was in Madrid four hours ago and is currently in Gothenburg? This problem of defining a multitrajectory PDF to answer trajectory related questions has received little attention in the MTT literature and, in this section, we present arguments that support the idea that such a multitrajectory PDF should be defined on the space of sets of trajectories. To this end, we proceed to review some characteristics of labelled RFS approaches to MTT and indicate their shortcomings.

In the typical labelled RFS approach, we only consider multitarget PDFs on the set of labelled targets at a certain time. Importantly, the sequence of these multitarget PDFs does not contain all available information and does not let us answer trajectory related questions [27], which are of key importance in tracking. This is particularly significant in situations with labelling uncertainties, in which there is an unknown association between (unlabelled) targets states and labels. In this case, it is even complicated to construct sensible trajectory estimates [18] as discussed next.

With this approach, we cannot use optimal Bayesian trajectory estimators because we do not have access to the multitrajectory PDF. A common strategy is instead to follow a two-step track building procedure: the set of labelled targets at a time step is estimated from its multitarget PDF and then trajectories are built by linking target state estimates with the same label across time. This procedure works well in many cases despite not being Bayesian optimal. However, it is problematic in some cases and post-hoc fixes are required if sensible estimates are to be obtained. As will be explained in the following, two examples of such situations are when the birth process is an IID cluster RFS, in which the new born targets are IID given the cardinality [10, Sec. 4.3.2], and when targets get in close proximity and then separate. This is an important weakness since an optimal Bayesian MTT algorithm should be able to handle all types of target birth processes and

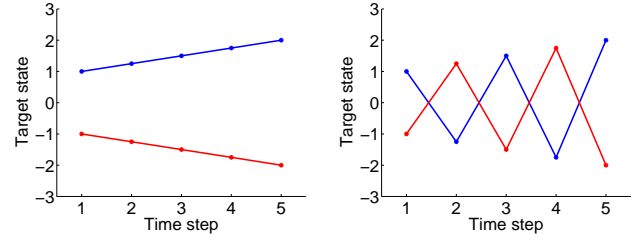


Figure 2: Illustration of two trajectory estimates from the sequence of labelled multitarget filtering/smoothing PDFs of Example 2. If we use the multitrajectory filtering PDF (defined on the space of sets of trajectories), the only possible estimate is the one in the left figure.

interactions among targets. We illustrate the problem of the IID cluster birth RFS in a simple example.

Example 2. Let us consider the following scenario. New born targets at time 1 are modelled by a labelled multi-Bernoulli RFS [13] with two components whose labels are $a = (1, 1)$ and $b = (1, 2)$. Both components have existence probability one and an identical PDF, which can be any. Note that if we remove the labels, the new born targets are an IID cluster RFS. Targets move independently with a given transition density and probability of survival one and no more targets can be born afterwards. Target states (without labels) are observed using the standard measurement model with no clutter, probability of detection 1 and negligible noise [10]. At each time $k \in \{1, \dots, 5\}$, the measurement set is $\{z_1^k, z_2^k\}$.

The multitarget filtering PDF $\pi^k(\cdot)$ at time k , which is given by the δ -generalised labelled multi-Bernoulli (δ -GLMB) filter [13] and coincides with the smoothing solution, is

$$\begin{aligned} \pi^k(\{(x_1^k, l_1^k), (x_2^k, l_2^k)\}) \\ = \frac{1}{2} (\delta(x_1^k - z_1^k) \delta(x_2^k - z_2^k) + \delta(x_2^k - z_1^k) \delta(x_1^k - z_2^k)) \\ \times (\delta[l_1^k - a] \delta[l_2^k - b] + \delta[l_2^k - a] \delta[l_1^k - b]) \end{aligned} \quad (2)$$

and zero for other sets of labelled targets. Notation $\delta(\cdot)$ and $\delta[\cdot]$ represent the Dirac and Kronecker delta, respectively, and (x_j^k, l_j^k) represents the state and label of target j at time k . Evidently, the multitarget filtering/smoothing PDF at any time k contains the information that there are targets located at $\{z_1^k, z_2^k\}$, but labelling them as $\{(z_1^k, a), (z_2^k, b)\}$ is as likely as $\{(z_1^k, b), (z_2^k, a)\}$. As a result, the filtering/smoothing multitarget PDFs in isolation cannot tell us how to link target state estimates at different time steps, which can lead to track switching, as illustrated in Figure 2.

It is true that, in practice, we can develop pragmatic fixes to obtain sensible trajectory estimates in this example. However, these fixes are not theoretically correct and therefore hamper the power of adopting a full Bayesian methodology. \square

The track switching problem illustrated in Example 2 arises in general for sets of labelled targets when new born targets are an IID cluster RFS [17]. In this case, track switching never stops as in Figure 2. This is due to the fact that any labelling of the target state estimates is equally likely [18] as the multitarget filtering PDFs are permutation invariant [29]. Even if we use a different birth process, the problem still arises when targets get in close proximity and then separate, which

is a scenario that has been studied thoroughly in the literature [17], [18]. We will further analyse the track switching problem of the δ -GLMB filter via simulations in Section V, e.g., see Figure 6.

The issues of track switching and the impossibility of answering trajectory related questions could be resolved by adopting the sequence of sets of labelled targets as the state variable. However, due to the inclusion of arbitrary labels, this state does not uniquely represent the underlying physical reality, see Example 1, which implies that we cannot define metrics with physical interpretation. The problem is that, due to the identity property of metrics [8], we always obtain a non-zero distance/error between two different sequences of sets of labelled targets describing the same trajectories, see Figure 1, independently of the metric we use. Metrics are relevant for performance evaluation and optimal estimation as they are involved in the concept of miss-distance, or error, between a reference variable and its estimate [8]. Due to this lack of metric, we cannot evaluate algorithms based on sequence of sets of labelled targets in a mathematically sound manner.

In short, in this section, we have highlighted two important characteristics that a Bayesian MTT solution should have: availability of a multitrajectory PDF to answer trajectory related questions and a metric on the underlying state space for evaluating algorithms and obtaining optimal estimators. Labelled approaches cannot handle both. On the contrary, sets of trajectories do not include arbitrary parameters so we can develop metrics, such as the one proposed in [23], to assess algorithms and develop optimal estimators. In addition, a (multioject) PDF on the set of trajectories enables us to answer all possible trajectory related questions so it satisfactorily overcomes the problems of labelled representations. Therefore, we argue that the objective of Bayesian MTT is to compute the multitrajectory PDF over the set of trajectories.

A convenient way to obtain this desired multitrajectory PDF is to use the set of trajectories as state variable in the filter. There are also additional benefits of adopting the sets of trajectories as the state variable. For example, it enables the derivation of intrinsically unlabelled filters to build trajectories, such as the trajectory PHD filter [24]. There are also algorithmic advantages compared to a sequence of labelled sets, for example, a decrease in the number of parameters and the availability of closed-form filtering formulas for the standard measurement model. These formulas are not available in the literature for a sequence of labelled sets but we provide them for sets of trajectories in Section III-B.

C. Probability and integration

In this paper, probability and integration are defined using finite set statistics (FISST) [9], [10], which is related to measure theory [26]. Even though FISST usually considers sets of targets, it can be applied to sets of trajectories by changing the single object state (targets by trajectories) and single object integrals (single target integrals by single trajectory integrals). Therefore, given a real-valued function $\pi(\cdot)$ on the space $\mathcal{F}(T)$ of sets of trajectories, its set integral is [9]:

$$\int \pi(\mathbf{X}) \delta \mathbf{X}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \pi(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(t_{1:n}, i_{1:n}) \in \mathbb{N}^{2n}} \int \pi(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) \\ &\quad \times d(x_1^{1:i_1}, \dots, x_n^{1:i_n}) \end{aligned} \quad (3)$$

where $X_{1:n} = (X_1, \dots, X_n)$. The integral goes through all possible cardinalities, start times, lengths and target states of the trajectories. Note that, if $\pi(\cdot)$ is a multitrajectory PDF, then, $\pi(\cdot) \geq 0$ and its set integral is one.

We can also use set integrals to calculate the probability that an RFS of trajectories belongs to a certain region. In order to do so, we define a mapping $\chi : \mathfrak{U}_{n=0}^{\infty} T^n \rightarrow \mathcal{F}(T)$ of sequences of trajectories to sets of trajectories such that $\chi((X_1, \dots, X_n)) = \{X_1, \dots, X_n\}$. Given a region $A = \mathfrak{U}_{n=0}^{\infty} A_n$ where $A_n \subseteq \mathcal{F}(T^n)$ is a set that contains sets with n elements in T , the probability that \mathbf{X} belongs to A is

$$P(\mathbf{X} \in A) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\chi^{-1}(A_n)} \pi(\{X_1, \dots, X_n\}) dX_{1:n}. \quad (4)$$

where $\pi(\cdot)$ is the multitrajectory PDF of \mathbf{X} . Equation (4) is proved in Appendix A. For instance, if $A_2 = \{\{X_1, X_2\} : X_1 \in B_1, X_2 \in B_2\}$ where $B_1 \subseteq T$ and $B_2 \subseteq T$, then $\chi^{-1}(A_2) = \{(X_1, X_2) : X_1 \in B_1, X_2 \in B_2 \text{ or } X_2 \in B_1, X_1 \in B_2\}$. Calculating (4) for A_2 we obtain the probability that there are two trajectories, one in region B_1 and another one in region B_2 . Equation (4) is necessary to obtain certain probabilities of interest, for example, the probability that there is a number of trajectories present at a certain time instant or a number of targets in a given region at a given time. The cardinality distribution $\rho(\cdot)$ indicates the probability that n trajectories have existed at all times

$$\begin{aligned} \rho(n) &= P(\mathbf{X} \in \chi(T^n)) \\ &= \frac{1}{n!} \int \pi(\{X_1, \dots, X_n\}) dX_{1:n}, \end{aligned} \quad (5)$$

which is analogous for RFSs of targets [9, Eq. (11.115)].

Example 3. We consider a multitrajectory PDF $\pi(\cdot)$ such that

$$\begin{aligned} \pi(\{(1, x_1^{1:2})\}) &= 0.9 \mathcal{N}\left(x_1^{1:2}; (10, 11), \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) \\ \pi(\{(1, x_1^{1:2}), (2, x_2^1)\}) &= 0.1 \mathcal{N}\left(x_1^{1:2}; (10, 11), \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) \\ &\quad \times \mathcal{N}(x_2^1; 100, 1), \end{aligned}$$

where $\mathcal{N}(\cdot; \bar{x}, P)$ denotes a Gaussian PDF with mean \bar{x} and covariance matrix P , and $\pi(\cdot)$ is zero for other sets of trajectories. From (5), we see that the probability that there is one trajectory is 0.9 and the probability for two trajectories is 0.1. The probability that there is only one trajectory and this trajectory starts at time step 1 in a region $B_1 \subseteq D$ and moves to a region $B_2 \subseteq D$ at the next time step can be obtained by integrating $\pi(\cdot)$ over the region $A = \chi(\{1\} \times B \times \mathfrak{U}_{i=0}^{\infty} D^i)$, where $B = B_1 \times B_2$. That is, region A considers start time 1 with the first two states belonging to B . Then, the trajectory

can die at any moment afterwards and, when it is present, its state at a particular time belongs to D . Then,

$$P(\mathbf{X} \in A) = 0.9 \int_B \mathcal{N}\left(x_1^{1:2}; (10, 11), \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) dx_1^{1:2}$$

where we have used that $\chi^{-1}(A_1) = \{1\} \times B \times \uplus_{i=0}^{\infty} D^i$. \square

III. FILTERING RECURSION FOR RFSS OF TRAJECTORIES

In this section, we present the filtering recursion for RFS of trajectories. We consider the conventional dynamic assumptions for MTT used in the RFS framework [9]: at each time step, a target follows a Markovian process such that it survives with a probability $p_S(\cdot)$ and moves with a transition PDF $g(\cdot|\cdot)$ and new born targets have a multitarget PDF $\beta_\tau(\cdot)$, where multitarget PDFs have the subindex τ to differentiate them from multitrajectory PDFs. Note that, as in filtering RFS of targets, this model implies that the number of trajectories and new born targets at each time step is unknown. As we will see, this dynamic model gives rise to a transition multitrajectory PDF $f^k(\cdot|\cdot)$ for the set of trajectories at time k , which includes all trajectories that have ever been present.

Example 4. We proceed to illustrate how the set of trajectories of Example 1, which is represented in Figure 1, evolves with time. We consider that the current time step is $k = 5$ and discuss what the set may look like at time $k = 6$. Trajectory X_1 , which is not present at time 5, remains unaltered. Trajectory X_2 survives with probability $p_S(1)$, which means that it becomes $X_2 = (1, (0.5, 0.625, 0.75, 0.875, 1, y))$ with $y \sim g(\cdot|1)$, or remains unaltered with probability $1 - p_S(1)$. An analogous behaviour is shown by X_3 . The new set is guaranteed to contain these three trajectories plus new trajectories determined by the new born targets, generated from the birth process $\beta_\tau(\cdot)$, and the time of appearance 6. \square

For this dynamic model, we present the filtering recursion for a general measurement model in Section III-A and for the standard measurement model in Section III-B.

A. General measurement model

The set of targets at time k is observed through noisy measurements giving rise to the likelihood $\ell^k(\cdot)$, where we omit the value of the measurement for notational simplicity [30]. By using FISST [10], the multitrajectory filtering PDF $\pi^k(\cdot)$ at time k , i.e., the PDF of the set of trajectories given the sequence of measurements up to time step k , can be calculated recursively using the prediction and update equations,

$$\pi^{k|k-1}(\mathbf{X}) = \int f^k(\mathbf{X}|\mathbf{Y}) \pi^{k-1}(\mathbf{Y}) \delta \mathbf{Y} \quad (6)$$

$$\pi^k(\mathbf{X}) \propto \ell^k(\mathbf{X}) \pi^{k|k-1}(\mathbf{X}). \quad (7)$$

Here, \propto means “is proportional to” and $\pi^{k|k-1}(\cdot)$ is referred to as the predicted multitrajectory PDF at time k , which represents the PDF of the set of trajectories at time k given the sequence of measurements up to time $k-1$. For general track-before-detect measurement models, the likelihood $\ell^k(\cdot)$ cannot be simplified so we just write the update as in (7) [30]. The following theorem indicates how to evaluate the prediction (6)

more explicitly for the conventional dynamic model, explained at the beginning of Section III.

Theorem 5. Given a set \mathbf{W} of new born trajectories at time k , a set \mathbf{X} of trajectories present at times $k-1$ and k but not present at $k+1$, a set \mathbf{Y} of trajectories present at time $k-1$ but not present at time k and a set \mathbf{Z} of trajectories present at a time before $k-1$ but not at $k-1$, the predicted multitrajectory PDF $\pi^{k|k-1}(\cdot)$ at time k is

$$\begin{aligned} \pi^{k|k-1}(\mathbf{W} \uplus \mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}) &= \prod_{(t, x^{1:i}) \in \mathbf{X}} (g(x^i | x^{i-1}) p_S(x^{i-1})) \prod_{(t, x^{1:i}) \in \mathbf{Y}} (1 - p_S(x^i)) \\ &\times \pi^{k-1}(\cup_{(t, x^{1:i}) \in \mathbf{X}} \{(t, x^{1:i-1})\} \uplus \mathbf{Y} \uplus \mathbf{Z}) \beta_\tau(\tau^k(\mathbf{W})). \end{aligned}$$

Theorem 5 is proved in Appendix B. We first clarify that if $(t, x^{1:i}) \in \mathbf{W}$, then, $t = k$, $i = 1$; if it belongs to \mathbf{X} , then $t < k$, $i = k - t + 1$; if it belongs to \mathbf{Y} , then $t < k$, $i = k - t$; and finally, if it belongs to \mathbf{Z} , then, $t < k - 1$, $i < k - t$. To evaluate the predicted multitrajectory PDF at time k , we multiply the following terms: multitrajectory filtering PDF $\pi^{k-1}(\cdot)$ for trajectories present at previous times, $g(\cdot|\cdot)$ and $p_S(\cdot)$ for surviving trajectories, $(1 - p_S(\cdot))$ for trajectories present at time $k-1$ but not present at k and the multitarget PDF for new born targets.

B. Standard measurement model

In this section we present the following key result: for the standard measurement model (8) and the birth model (9), the multitrajectory filtering and predicted PDFs at all time steps have the same form. Therefore, this type of multitrajectory PDF is conjugate with respect to the standard measurement likelihood [13]. How to obtain these multitrajectory PDFs recursively is indicated by Lemmas 7 and 8.

We use the multiobject exponential notation $h^\mathbf{x} = \prod_{x \in \mathbf{x}} h(x)$ where h is a real-valued function and $h^\emptyset = 1$ by convention [13]. For a given multi-target state \mathbf{x} at time k , each target state $x \in \mathbf{x}$ is either detected with probability $p_D(x)$ and generates a measurement with PDF $l(z|x)$, or missed with probability $1 - p_D(x)$. The observation is the superposition of the target-generated measurements and Poisson clutter with intensity function $\kappa(\cdot)$. Given a set $\mathbf{X} = \{X_1, \dots, X_n\}$ with n present trajectories at time k , a set \mathbf{Y} of trajectories with no present trajectories at time k and the measurement set $\mathbf{z}^k = \{z_1^k, \dots, z_m^k\}$, the likelihood at time k is

$$\ell^k(\mathbf{X} \uplus \mathbf{Y}) = e^{-\int \kappa(z) dz} \kappa^{\mathbf{z}^k} \sum_{\theta \in \Theta_{n,m}} \prod_{j=1}^n \psi_{\mathbf{z}^k}(X_j | \theta_j), \quad (8)$$

where $\Theta_{n,m}$ denotes all data association hypotheses for n targets and m measurements, $\theta = (\theta_1, \dots, \theta_n)$ with $\theta_i = j$ if the i th present target is associated with the j th measurement or 0 if it is undetected and

$$\psi_{\mathbf{z}^k}(t, x^{1:i} | \theta_j) = \begin{cases} \frac{p_D(x^i) l(z_{\theta_j}^k | x^i)}{\kappa(z_{\theta_j}^k)} & \theta_j > 0 \\ 1 - p_D(x^i) & \theta_j = 0. \end{cases}$$

If $\mathbf{X} = \emptyset$, then $\ell^k(\mathbf{Y}) = e^{-\int \kappa(z) dz} \kappa^{\mathbf{z}^k}$.

In order to obtain the explicit recursion, we assume that the targets can be born from b PDFs $\beta_1(\cdot), \dots, \beta_b(\cdot)$ with a weight $w_B^k(L) : L \subseteq \mathbb{N}_b$, which indicates the probability that $|L|$ targets are born from the PDFs indicated by L . The resulting multitarget PDF is

$$\beta_\tau(\{x_1, \dots, x_n\}) = \sum_{l_{1:n}}^{\neq} w_B(\{l_1, \dots, l_n\}) \prod_{j=1}^n \beta_{l_j}(x_j), \quad (9)$$

where $n \leq b$ and the sum is performed over distinct elements:

$$\sum_{l_{1:n}}^{\neq} = \sum_{l_{1:n}: l_1 \neq \dots \neq l_n}.$$

Note that we can draw samples from (9) by first drawing the auxiliary set L , whose cardinality is the number of new born targets, from $w_B(\cdot)$ and then drawing the target states from the corresponding PDFs independently.

Example 6. Let us consider one-dimensional targets that are born according to the model (9) with $b = 2$,

$$\beta_i(x) = \mathcal{N}(x; \mu_i, \sigma_i^2)$$

where μ_i and σ_i^2 represent the mean and variance of the i th birth component, $w_B(\emptyset) = 0.8$, $w_B(\{1\}) = 0.1$, $w_B(\{2\}) = 0.05$ and $w_B(\{1, 2\}) = 0.05$. This means that no target is born with probability 0.8, one target is born from density $\beta_1(\cdot)$ with probability 0.1 and from $\beta_2(\cdot)$ with probability 0.05. Two targets are born with probability 0.05, one from density $\beta_1(\cdot)$ and another from $\beta_2(\cdot)$. \square

If the likelihood is (8) and the multitarget PDF of new born targets is (9), we show in this section that the multitarget filtering PDF can be written as

$$\begin{aligned} \pi^k(\{X_1, \dots, X_n\}) &= \sum_{h_{1:n}^{k|k}}^{\neq} w^{k|k}(\{h_1^{k|k}, \dots, h_n^{k|k}\}) \\ &\quad \times \prod_{j=1}^n p^{k|k}(X_j | h_j^{k|k}) \end{aligned} \quad (10)$$

where $h_j^{k|k} = (l_j, t_j, i_j, \xi_j)$ is a single trajectory hypothesis and implies that the density $p^{k|k}(\cdot | h_j^{k|k})$ (on T) has been obtained by propagating birth component $\beta_{l_j}(\cdot)$ with starting time t_j , duration i_j and data associations ξ_j , which is a vector of length i_j that takes values 0 if the density is associated with clutter or i if it is associated with the i th measurement at the corresponding time step. We can see that hypothesis $h_j^{k|k}$ includes the pair (l_j, t_j) , which corresponds to the label in [13], but the label is not included in the trajectory state. More details about the labelled approach are given in Section IV-A.

We also show that the predicted multitarget PDF at time k has the same form as (10) and can be written as

$$\begin{aligned} \pi^{k|k-1}(\{X_1, \dots, X_n\}) &= \sum_{h_{1:n}^{k|k-1}}^{\neq} w^{k|k-1}(\{h_1^{k|k-1}, \dots, h_n^{k|k-1}\}) \\ &\quad \times \prod_{j=1}^n p^{k|k-1}(X_j | h_j^{k|k-1}) \end{aligned} \quad (11)$$

where $h_j^{k|k-1}$ is the same as $h_j^{k|k}$ but the data association vector has $i_j - 1$ components as the data association at time k has not been done yet. The resulting steps of the recursion, which are explained in the rest of the section, are shown in Algorithm 1. As we consider the beginning of the simulation at time step 1, see Section II-A, we can set $w^{0|0}(\emptyset) = 1$ such that trajectories at time 1 are born according to the birth model. In addition, $w^{0|0}(\emptyset) = 1$ is a particular case of (10) so the type of multitarget PDF (10) is conjugate for the standard model.

Algorithm 1 Calculation of multitarget filtering PDF for the standard measurement model

```

- Initialisation:  $w^{0|0}(\emptyset) = 1$ .
for  $k = 1$  to final time step do
  - Prediction: Generate the new hypothesis sets and calculate/approximate their weights  $w^{k|k-1}(\cdot)$  and trajectory PDFs  $p^{k|k-1}(\cdot)$  using Lemma 7.
  - Update: Generate the new hypothesis sets and calculate/approximate their weights  $w^{k|k}(\cdot)$  and trajectory PDFs  $p^{k|k}(\cdot)$  using Lemma 8.
end for
```

Before providing the recursive formulas for computing (10) and (11), we introduce the following sets of single trajectory hypotheses: \mathbb{U}^k contains the hypotheses of a surviving trajectory at time k that has a data association hypothesis at time k ; \mathbb{S}^k contains the hypotheses of a surviving trajectory at time k that does not yet contain a data association hypothesis at time k ; \mathbb{D}^k contains the hypotheses of trajectories present at time $k - 1$ but not present at time k ; \mathbb{N}^k contains the hypotheses of new born trajectories at time k and $\mathbb{D}^{1:k} = \uplus_{j=1}^k \mathbb{D}^j$, which considers trajectories that ended at time $k - 1$ or earlier. After the k th update step, a single trajectory hypothesis is contained in $\mathbb{U}^k \uplus \mathbb{D}^{1:k}$. Before the k th update step, a single trajectory hypothesis is contained in $\mathbb{S}^k \uplus \mathbb{N}^k \uplus \mathbb{D}^{1:k}$. Mathematically, these sets are given by

$$\begin{aligned} \mathbb{U}^k &= \{(l, t, i, \xi) : l \in \mathbb{N}_b, t \leq k, t + i - 1 = k, d(\xi) = i\} \\ \mathbb{S}^k &= \{(l, t, i, \xi) : l \in \mathbb{N}_b, t \leq k, t + i - 1 = k, d(\xi) = i - 1\} \\ \mathbb{D}^k &= \{(l, t, i, \xi) : l \in \mathbb{N}_b, t < k, t + i = k, d(\xi) = i\} \\ \mathbb{N}^k &= \{(l, k, 1) : l \in \mathbb{N}_b\} \end{aligned}$$

where $d(\xi)$ denotes the dimension of vector ξ .

Lemma 7 (Prediction). *Given $\pi^k(\cdot)$ of the form (10) and hypothesis sets $\mathcal{A} \subset \mathbb{S}^{k+1}$, $\mathcal{B} \subset \mathbb{D}^{k+1}$, $\mathcal{C} \subset \mathbb{N}^{k+1}$ and $\mathcal{D} \subset \mathbb{D}^{1:k}$, the predicted weight in (11) for hypothesis set $\mathcal{A} \uplus \mathcal{B} \uplus \mathcal{C} \uplus \mathcal{D}$ is*

$$\begin{aligned} w^{k+1|k}(\mathcal{A} \uplus \mathcal{B} \uplus \mathcal{C} \uplus \mathcal{D}) \\ = [\gamma_S]^{\mathcal{A}} [\gamma_D]^{\mathcal{B}} w_B^{k+1}(\mathcal{C}) w^{k|k}(\mathcal{A}^- \uplus \mathcal{B} \uplus \mathcal{D}) \end{aligned} \quad (12)$$

$$\gamma_S(h) = \int p_S(X^l) p^{k|k}(X|h^-) dX \quad (13)$$

$$\gamma_D(h) = 1 - \gamma_S(h) \quad (14)$$

$$w_B^k(\{(l_1, k, 1), \dots, (l_n, k, 1)\}) = w_B(\{l_1, \dots, l_n\}) \quad (15)$$

where $h = (l, t, i, \xi)$, $h^- = (l, t, i-1, \xi)$, $\mathcal{A}^- = \{h^- : h \in \mathcal{A}\}$ and X^l is the last target state of X .

The trajectory PDF for $h \in \mathbb{S}^{k+1} \uplus \mathbb{D}^{k+1} \uplus \mathbb{N}^{k+1} \uplus \mathbb{D}^{1:k}$ is

$$\begin{aligned} p^{k+1|k}(X|h) &= p_S^{k+1|k}(X|h) 1_{\mathbb{S}^{k+1}}(h) + p_D^{k+1|k}(X|h) 1_{\mathbb{D}^{k+1}}(h) \\ &\quad + p_N^{k+1|k}(X|h) 1_{\mathbb{N}^{k+1}}(h) + p^{k|k}(X|h) 1_{\mathbb{D}^{1:k}}(h) \end{aligned} \quad (16)$$

where

$$\begin{aligned} p_S^{k+1|k}(t, x^{1:i}|h) &= g(x^i | x^{i-1}) p_S(x^{i-1}) \\ &\quad \times p^{k|k}(t, x^{1:i-1}|h^-) / \gamma_S(h) \\ p_D^{k+1|k}(t, x^{1:i}|h) &= p^{k|k}(t, x^{1:i}|h) (1 - p_S(x^i)) / \gamma_D(h) \\ p_N^{k+1|k}(t, x^1|l, t, 1) &= \beta_l(x^1) \delta[t - (k+1)]. \end{aligned}$$

The proof is straightforward by applying Theorem 5 to (10). Each hypothesis set in (12) can be decomposed into disjoint hypothesis sets \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} that describe surviving trajectories, present trajectories at time k but not at time $k+1$, new born trajectories and trajectories that were present some time before time k but not at time k , respectively. The weight $w^{k+1|k}(\cdot)$ corresponds to the weight $w^{k|k}(\cdot)$ of the parent hypothesis set $\mathcal{A}^- \uplus \mathcal{B} \uplus \mathcal{D}$ multiplied by the weight of the hypothesis set \mathcal{C} of new born targets, the probability (13) of survival for trajectories hypothesised in \mathcal{A} and the probability (14) of death for trajectories hypothesised in \mathcal{B} . The resulting PDF of a trajectory given a hypothesis is given by (16). Densities $p_S^{k+1|k}(\cdot|h)$, $p_D^{k+1|k}(\cdot|h)$ and $p_N^{k+1|k}(\cdot|h)$ correspond to a surviving trajectory, a trajectory present at time k but not at time $k+1$, and a new born trajectory, respectively. If the trajectory is not present at time k , which means that its hypothesis is contained in $\mathbb{D}^{1:k}$, its PDF remains unaltered.

Lemma 8 (Update). Given $\pi^{k|k-1}(\cdot)$ of the form (11), the measurement set $\mathbf{z}^k = \{z_1^k, \dots, z_m^k\}$ and hypothesis sets $\mathcal{D} \subset \mathbb{D}^{1:k}$ and $\mathcal{E} = \{h_1^{k|k}, \dots, h_n^{k|k}\} \subset \mathbb{U}^k$ such that $h_j^{k|k} = (h_j^{k|k-1}, \xi_j)$, $(\xi_1, \dots, \xi_n) \in \Theta_{n,m}$, the filtering weight in (10) for hypothesis set $\mathcal{E} \uplus \mathcal{D}$ is

$$w^{k|k}(\mathcal{E} \uplus \mathcal{D}) \propto w^{k|k-1}(\mathcal{E}^\circ \uplus \mathcal{D}) [\eta_{\mathbf{z}^k}]^\mathcal{E} \quad (17)$$

$$\eta_{\mathbf{z}^k}(h_j^{k|k-1}, \xi_j) = \int \psi_{\mathbf{z}^k}(X|\xi_j) p^{k|k-1}(X|h_j^{k|k-1}) dX \quad (18)$$

where $\mathcal{E}^\circ = \{h_1^{k|k-1}, \dots, h_n^{k|k-1}\}$.

The trajectory PDF for $h_j^{k|k} \in \mathbb{U}^k \uplus \mathbb{D}^{1:k}$ is

$$\begin{aligned} p^{k|k}(X|h_j^{k|k}) &= p_U^{k|k}(X|h_j^{k|k}) 1_{\mathbb{U}^k}(h_j^{k|k}) \\ &\quad + p^{k|k-1}(X|h_j^{k|k}) 1_{\mathbb{D}^{1:k}}(h_j^{k|k}) \end{aligned} \quad (19)$$

$$p_U^{k|k}(X|h_j^{k|k-1}, \xi_j) = \frac{\psi_{\mathbf{z}^k}(X|\xi_j) p^{k|k-1}(X|h_j^{k|k-1})}{\eta_{\mathbf{z}^k}(h_j^{k|k-1}, \xi_j)}.$$

The proof is straightforward by applying Bayes' rule (7) to each component of the predicted multitrajectory PDF and data association. A hypothesis set in (17) can be decomposed into

disjoint hypothesis sets $\mathcal{E} \uplus \mathcal{D}$ that describe present trajectories at time k and trajectories that were present before time k but not at time k , respectively. The weight $w^{k|k}(\cdot)$ corresponds to the weight $w^{k|k-1}(\cdot)$ of the parent hypothesis $\mathcal{E}^\circ \uplus \mathcal{D}$ multiplied by the data association probabilities $\eta_{\mathbf{z}^k}(\cdot)$ of the present trajectories. The resulting PDF of a trajectory for a hypothesis is given by (19). Density $p_U^{k|k}(\cdot|h_j^{k|k})$ corresponds to a present trajectory at time k . If the trajectory is not present at time k , which means that its hypothesis is contained in $\mathbb{D}^{1:k}$, its PDF remains unaltered.

Even though the presented recursion can be called closed-form in an RFS context, as in [13], [14], approximations are required in general. As happens in single target applications, we need to perform single trajectory PDF approximations to calculate (16) and (19) for non-linear/non-Gaussian dynamic/measurements models. In fact, it is important to highlight is that the update and prediction of the weights only depend on the PDFs of the target states at the current time. In other words, $\gamma_S(\cdot)$, $\gamma_D(\cdot)$ and $\eta_{\mathbf{z}^k}(\cdot)$ are calculated/approximated using a single target integral w.r.t. the density of the target $\tau^k(X)$ for the corresponding hypothesis.

In addition, if we run the filter for a sufficiently long time, the filter is intractable to implement due to the ever increasing dimensionality of the PDFs involved and number of hypotheses. A practical solution for the trajectory PHD filter, which can also be adapted for the general equations in Lemmas 7 and 8, can be found in [24].

1) *On the number of hypotheses:* In this section, we indicate how the number of hypotheses grows in the prediction and update steps. Given a prediction hypothesis set with n present trajectories and the number m of measurements at time k , the number of hypothesis sets generated from this prediction hypothesis set in the update step is

$$\sum_{j=0}^{\min(m,n)} j! c_j^m c_j^n \quad (20)$$

where j represents the number of detected targets and c_j^m is a binomial coefficient, see Nomenclature. Given the number of detected targets, the number of possible measurement to target associations is $j! c_j^m c_j^n$.

Given an updated hypothesis set with n present trajectories, the number of possible hypothesis sets generated from it in the prediction step is 2^{n+b} , where b is the maximum number of targets that can be born at a time. Each of the $n+b$ possible trajectories is either present or not.

C. IID cluster birth process

In this section, we analyse the case in which new born targets are an IID cluster RFS with the standard measurement model described in Section III-B. We conclude that labels are not necessary in the hypotheses and we can lower the number of hypotheses by not using them. If new born targets are an IID cluster RFS, $\beta_j(\cdot) = \beta_i(\cdot)$ for $i, j \in \{1, \dots, b\}$. In this particular case, all previous equations are completely valid even though the index on the birth PDF, represented by l in the hypotheses, is meaningless. Nevertheless, as new born

target PDFs are alike, we can remove l from the hypotheses to lower the number of hypotheses. We denote a hypothesis without initial birth component as $\hat{h}_j^{k|k} = (t_j, i_j, \xi_j)$. Then, it is shown in Appendix C that the multitrajectory filtering PDF can be written as

$$\pi^k(\{X_1, \dots, X_n\}) = \sum_{\hat{h}_{1:n}^{k|k}} \hat{w}^{k|k}(\hat{h}_{1:n}^{k|k}) \prod_{j=1}^n p^{k|k}(X_j | \hat{h}_j^{k|k}) \quad (21)$$

where hypotheses are now sequences $\hat{h}_{1:n}^{k|k} = (\hat{h}_1^{k|k}, \dots, \hat{h}_n^{k|k})$ and $\hat{w}^{k|k}(\cdot)$ is the weight of the hypothesis. It should be noted that $\hat{h}_{1:n}^{k|k}$ can have repeated elements and $\hat{w}^{k|k}(\cdot)$ is permutation invariant so we only need to compute one weight for any ordering of the components in $\hat{h}_{1:n}^{k|k}$. As indicated in Section III-C1, this results in a lower number of hypotheses than in the labelled case. This kind of simplification is not possible if we use sequences of labelled sets or labelled sets of trajectories, which are defined in Section IV-A, as state variables. Clearly, $\pi^{k|k-1}(\cdot)$ can also be written as (21) and we can obtain recursive formulas for $\hat{w}^{k|k}(\cdot)$ and $p^{k|k}(\cdot | \hat{h}_j^{k|k})$ as in Lemmas 7 and 8.

1) *On the number of hypotheses:* In this case, in the update step, the number of generated hypotheses from a single hypothesis is the same as before, see (20). However, given an updated hypothesis set with n present trajectories, the number of possible different hypothesis sets generated in the prediction step is $(b+1)2^n$. The hypotheses for new born targets basically correspond to the number of new born targets as all of them are IID. We can see that $(b+1)2^n \leq 2^{n+b}$ so the number of hypotheses is lower even though the final expressions (10) and (21) are alike.

IV. RELATION WITH OTHER MTT MODELS

In this section we relate the proposed filtering based on set of trajectories with other labelled and unlabelled RFS models and classical MHT.

A. Relation with sets of labelled trajectories

In this section we analyse the labelling of the trajectories. In [13], a target label is the pair (t, l) , where t is the birth time and l is the component of the birth model (9) from which the target is generated. A labelled target state corresponds to adding this (unique) label to its state. We can easily extend this label assignment for trajectories such that a labelled trajectory state is formed as $(l, t, x^{1:i})$ where l indicates that it was born from component l of the birth model, see (9). In this case, the calculation of the multitrajectory filtering PDF $\pi_l^k(\cdot)$ is analogous to what was presented previously in Section III-B:

$$\begin{aligned} \pi_l^k(\{(l'_1, X_1), \dots, (l'_n, X_n)\}) \\ = \sum_{\hat{h}_{1:n}^{k|k}} \hat{w}^{k|k}(\{\hat{h}_1^{k|k}, \dots, \hat{h}_n^{k|k}\}) \prod_{j=1}^n p_l^{k|k}((l'_j, X_j) | \hat{h}_j^{k|k}) \end{aligned} \quad (22)$$

where

$$p_l^{k|k}((l', X) | l, t, i, \xi) = \delta[l' - l] p^{k|k}(X | l, t, i, \xi). \quad (23)$$

That is, the labelled multitrajectory filtering PDF simply consists of adding a Kronecker delta determined by the initial birth component l in the trajectory hypothesis. Note that the labelled set of targets at time k can be simply obtained by obtaining the corresponding target state at time k and its label from the labelled trajectories, as was done in (1). Equivalently, there is a mapping from the labelled set of trajectories up to time step k to a sequence of labelled sets [14] so we can obtain the same information from both representations. However, to our knowledge, a closed-form formula such as (22) has not been proposed yet for sequence of labelled sets.

In the case of sets of trajectories and the standard measurement model, analysed in Section III-B, we can consider different kinds of hypotheses depending on the birth model. If we use the birth model (9), labels appear naturally in the hypothesis set. Labels indicate the start time and component of the birth model PDF (9) where a trajectory PDF of hypothesis is originated. In some cases, labels can represent that a trajectory originated at a certain time at one of the possible birth locations, e.g., the airport where an airplane departs from (assuming that only one airplane can depart from an airport at a time). Nevertheless, the labels in (9)-(11) are hidden variables as the information of where a trajectory started is already included in the trajectory itself.

If the birth process is an IID cluster process, the initial birth component variables become redundant and can be omitted. This also has the benefit of lowering the number of hypotheses, as pointed out in [31]. On the contrary, the labelled approach would still include them in the target state which prevents lowering the number of hypotheses.

B. Relation with MHT algorithms

Classical MHT algorithms [5], [31], [32] were developed for the standard measurement model but not for general track-before-detect models [30]. They rely on enumerating multiple target/measurement association hypotheses, calculating their probabilities and the PDF of the current target state given an hypothesis. For the standard measurement model, there are also algorithms based on RFS of targets that have this MHT-type structure [10], [13], [21]. Moreover, for this measurement model, the multitrajectory filtering PDF, which is given by (10), is actually also computed by considering multiple hypotheses but with PDFs over trajectories instead of targets. This is analogous to the algorithms in [33], which assume a single, always existing trajectory. In this sense, the proposed filtering equations for the standard measurement model can be considered a form of MHT algorithm, derived using sets of trajectories and FISST.

Compared to classical MHT and MHT-type algorithms for RFS of targets, we introduce the set of trajectories as a random variable that represents all quantities of interest. This enables us to compute the posterior distribution of the set of trajectories in a direct Bayesian manner and answer all types of questions regarding the history, as exemplified in Section

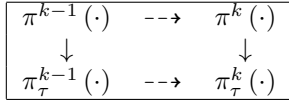


Figure 3: Relation between the multitarget filtering PDFs $\pi_\tau^{k-1}(\cdot)$ and $\pi_\tau^k(\cdot)$ and the multitrajectory filtering PDFs $\pi^{k-1}(\cdot)$ and $\pi^k(\cdot)$. Dashed arrows stand for prediction and update and the solid arrows for marginalisation.

II-B. Another advantage of introducing sets of trajectories is that we can use sound definitions of estimation error based on metrics [23] and optimal Bayesian estimators. In addition, compared to classical MHT algorithms, sets of trajectories can also be used with other measurement models, such as track-before-detect, or to develop new algorithms, such as the trajectory PHD filter [24], which do not give rise to an MHT-type structure. Nevertheless, the development of efficient MHT-type algorithms based on sets of trajectories should be inspired by the vast literature on classical MHT and MHT-type algorithms with RFS of targets.

C. Relation with sets of unlabelled targets

In this section, we relate the proposed approach and the RFS filtering model based on unlabelled targets. We first provide a theorem for obtaining the multitarget PDF on the set of targets at a particular time given a multitrajectory PDF. This process resembles marginalisation of PDFs in vector spaces [34].

Theorem 9. *Given the multitrajectory PDF $\pi(\cdot)$ of \mathbf{X} , the multitarget filtering PDF $\pi_\tau^k(\cdot)$ of $\tau^k(\mathbf{X})$ is given by*

$$\pi_\tau^k(\mathbf{y}) = \int \delta_{\tau^k(\mathbf{X})}(\mathbf{y}) \pi(\mathbf{X}) d\mathbf{X}$$

where

$$\delta_{\mathbf{z}}(\mathbf{y}) = \begin{cases} 0 & \text{if } |\mathbf{z}| \neq |\mathbf{y}| \\ 1 & \text{if } |\mathbf{z}| = |\mathbf{y}| = 0 \\ \sum_{\sigma \in \Gamma_n} \prod_{j=1}^n \delta_{z_{\sigma_j}}(y_j) & \text{if } \begin{cases} \mathbf{y} = \{y_1, \dots, y_n\} \\ \mathbf{z} = \{z_1, \dots, z_n\} \end{cases} \end{cases}$$

is a multitarget Dirac delta centered at \mathbf{z} [9, Eq. (11.124)] and Γ_n is the set that includes all the permutations of $(1, \dots, n)$.

Theorem 9 is proved in Appendix D. In the usual RFS filtering framework with unlabelled targets, we obtain the multitarget filtering PDF $\pi_\tau^k(\cdot)$ using the filtering PDF $\pi_\tau^{k-1}(\cdot)$ and the corresponding prediction and update equations [9]. In Appendix E, we prove that given the multitrajectory PDF $\pi^{k-1}(\cdot)$ and its corresponding multitarget PDF $\pi_\tau^{k-1}(\cdot)$ at time $k-1$, if we apply Theorem 9 to the output of the prediction and update steps, which are given by (6) and (7), as expected, we obtain the same multitarget filtering PDF for the set of targets at time k as in the usual RFS filtering framework. This is illustrated in Figure 3.

V. ILLUSTRATIVE EXAMPLE

This section illustrates multiple target tracking using sets of trajectories via simulations. We have implemented the equations in Algorithm 1 directly performing pruning with

100 hypotheses. We know from the literature on MHT or labelled RFS that there are more efficient ways to manage the exponential growth in hypotheses. However, how to develop efficient algorithms for sets of trajectories is beyond the scope of this paper.

We assume a target state $x \in \mathbb{R}^2$ that consists of position and velocity. We only consider position in a one-dimensional space so that we can visualise the results easily. The birth process has the parameters: $b = 2$, $\beta_1(x) = \beta_2(x) = \mathcal{N}(x; [0, 0]^T, \text{diag}(25, 1))$, $w_B(\emptyset) = 0.85$, $w_B(\{1\}) = w_B(\{2\}) = 0.05$, $w_B(\{1, 2\}) = 0.05$. The single-target dynamic process parameters are: $p_S = 0.9$, $g(x^i | x^{i-1}) = \mathcal{N}(x^i; Fx^{i-1}, Q)$ where

$$F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{10} \begin{pmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{pmatrix}.$$

The intensity function of the Poisson clutter is $\kappa(z) = 1.4 \cdot \frac{1}{20} 1_{(-10, 10)}(z)$, which means that clutter is uniformly distributed in $(-10, 10)$ and there is an average of 1.4 clutter measurement per scan. The target-generated measurements have the following parameters: $p_D = 0.95$, $l(z|x) = \mathcal{N}(z; Hx, R)$ where $H = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $R = 10^{-4}$.

We consider 22 time steps and observe the measurements shown in Figure 4(a). These measurements have been generated from the set of trajectories represented in Figure 4(b). Given these measurements and the model parameters indicated above, we calculate the multitrajectory filtering PDF using Lemmas 7 and 8. We represent the posterior mean of the trajectories for the six hypotheses with largest weights in Figure 5. The most likely situation is the one shown in Figure 5(a), in which there are three trajectories. This hypothesis accurately represents the set of trajectories from which measurements were generated, see Figure 4(b). In the second most likely situation, there is an extra trajectory that appears at time 12. The third hypothesis, is a variant of the previous with a different measurement association at time 14 for that trajectory. Note that there are two very close measurements at time 14 in the region where that target lies. The fourth hypothesis includes a new born target at time step 22. The fifth one considers a different data association at time 13 for the trajectory that appears at time 12. The sixth hypothesis breaks one of the trajectories and considers two trajectories instead of one.

In this set-up, it seems convenient to estimate the set of trajectories as the posterior mean of the hypothesis with highest weight, which is represented by Figure 5(a) and does not have track switching. We have also implemented the δ -GLMB filter, which is the closed form solution to the labelled RFS filtering problem [13], and its estimate is the posterior mean of the hypothesis with highest weight at each time step. Due to the IID cluster birth process, there is track switching at all time steps for the trajectories born at the same time in the δ -GLMB filter estimate, which is shown in Figure 2. This was previously indicated in Example 2 and is not a desirable situation that can be avoided by using the set of trajectories as state variable.

We now consider another scenario with the same parameters but with $\kappa(z) = \frac{1}{20} 1_{(-10, 10)}(z)$ and $R = 10^{-3}$, i.e., lower

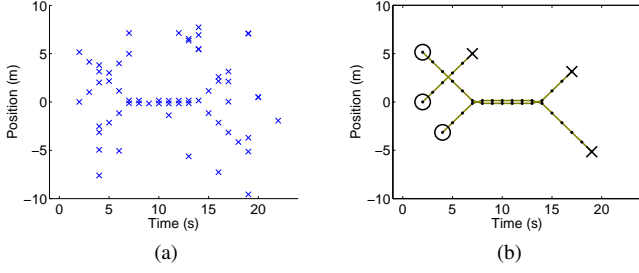


Figure 4: Scenario of the simulation: (a) Observed sequence of measurements. At time 14, there are two measurements close together at positions 5.45 and 5.47. (b) Trajectories from which the measurements have been generated. Circles and crosses indicate the start and end of trajectories.

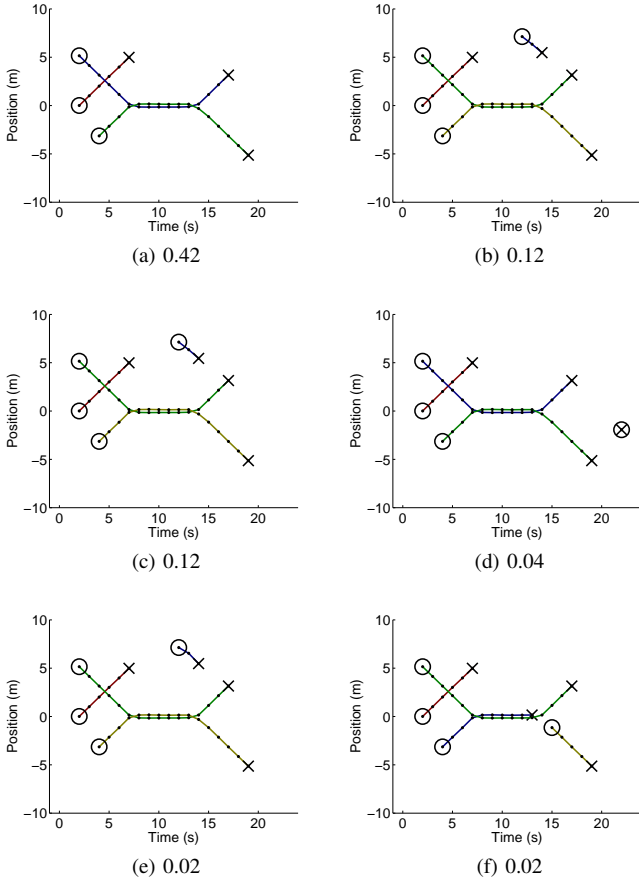


Figure 5: Posterior mean of the trajectories for the six hypotheses with largest weights, which are given in the subfigure captions.

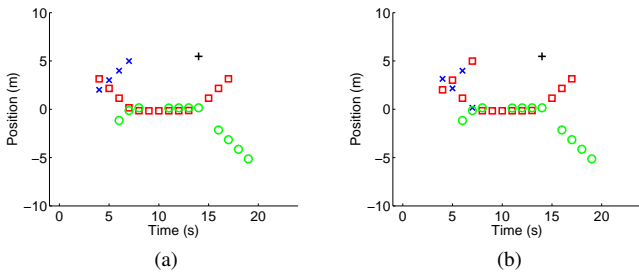


Figure 6: Two equally likely estimates provided by the δ -GLMB filter. Different markers represent different labels. Track formation using multitarget filtering PDFs does not work well.

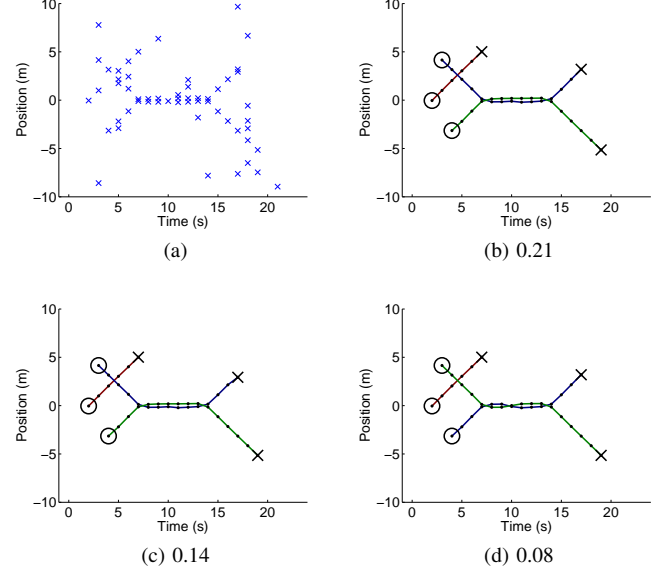


Figure 7: Measurements and three most likely hypotheses. Their weights are given in the subfigure captions. The third most likely hypothesis includes a trajectory switching.

clutter and higher measurement noise. We show the observed measurements and the three most likely hypotheses in Figure 7. None of the three most likely hypotheses has two trajectories that start at time 2 as the trajectories that were used to generate the measurements, see Figure 4(b). This is due to the fact that one of the trajectories has not been detected at time 2, see Figure 7(a). The third most likely hypothesis includes a trajectory switching when targets are in close proximity.

VI. CONCLUSIONS

In this paper we have proposed the set of trajectories as the state variable and have indicated how to use Mahler's RFS framework on this variable to perform Bayesian inference. This enables us to answer all trajectory related questions [27] from the multitrajectory PDF and define metrics with physical interpretation [23], which can be used to obtain optimal estimators and evaluate algorithms in a theoretically sound way. Therefore, sets of trajectories along with Mahler's RFS framework enable us to perform all the tasks of MTT in a Bayesian way, which cannot be handled with previously proposed labelled representations.

We have also derived a conjugate multitrajectory PDF used to obtain the multitrajectory filtering PDF for the standard measurement model. With this, we have clarified that labels correspond to hidden variables that appear in the hypothesis set but not in the state variable. If the birth process is an IID cluster, labels do not necessarily form part of the hypothesis set and we can lower the number of hypotheses by not using them. We have also provided relations with other labelled and unlabelled RFS models and illustrated via simulations how to perform MTT using sets of trajectories.

APPENDIX A

We explain the connection between multitarget PDFs and Janossy densities [35] and how to calculate probabilities.

A. Target case

A multitarget PDF packages the entire family of Janossy densities into a single function [36, p. 1161]. Given a multiobject PDF $\pi(\cdot)$, we can obtain the equivalent n th Janossy density $j_n(\cdot)$ as [10, Eq. (2.36)],

$$j_n(x_1, \dots, x_n) = \frac{1}{n!} \pi(\{x_1, \dots, x_n\}). \quad (24)$$

The Janossy density $j_n(\cdot)$ is the density of the n th Janossy measure w.r.t. the Lebesgue measure on D^n [35, p. 124]. Then,

$$j_n(x_1, \dots, x_n) dx_1 \dots dx_n \quad (25)$$

is the probability that there are exactly n points in the process, one in each of the n distinct infinitesimal regions $(x_i, x_i + dx_i) \subset D$ [35].

We define a region $A = \uplus_{n=0}^{\infty} A_n$ where $A_n \subseteq \mathcal{F}(D^n)$. That is, each A_n is a set that contains sets with n elements in D . Then, for \mathbf{x} distributed as $\pi(\cdot)$,

$$\begin{aligned} P(\mathbf{x} \in A) &= \sum_{n=0}^{\infty} P(\{x_1, \dots, x_n\} \in A_n) \\ &= \sum_{n=0}^{\infty} P(x_{1:n} \in \chi^{-1}(A_n)) \\ &= \sum_{n=0}^{\infty} \int_{\chi^{-1}(A_n)} j_n(x_{1:n}) dx_{1:n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\chi^{-1}(A_n)} \pi(\{x_1, \dots, x_n\}) dx_{1:n}. \end{aligned} \quad (26)$$

where $\chi(\cdot)$ denotes the mapping from vectors to sets, as indicated in Section II-C for trajectories or in [26].

B. Trajectory case

By analogy to (24)-(25), for a multitrajectory PDF $\pi(\cdot)$,

$$\frac{1}{n!} \pi(\{(t_1, x_1^{1:i_1}), \dots, (t_n, x_n^{1:i_n})\}) dx_1^{1:i_1} \dots dx_n^{1:i_n}$$

is the probability that there are exactly n distinct trajectories, with starting times and lengths $t_1, i_1, \dots, t_n, i_n$ in infinitesimal regions $(x_j^{1:i_j}, x_j^{1:i_j} + dx_j^{1:i_j}) \subset D^{i_j}$. For \mathbf{X} distributed as $\pi(\cdot)$ and a region $A = \uplus_{n=0}^{\infty} A_n$,

$$\begin{aligned} P(\mathbf{X} \in A) &= \sum_{n=0}^{\infty} P(\{X_1, \dots, X_n\} \in A_n) \\ &= \sum_{n=0}^{\infty} P(X_{1:n} \in \chi^{-1}(A_n)) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\chi^{-1}(A_n)} \pi(\{X_1, \dots, X_n\}) dX_{1:n}. \end{aligned}$$

APPENDIX B

In this appendix, we prove Theorem 5. First, we provide the transition density. Given $X = (t, x^{1:i})$ with $t + i - 1 < k$ and $Y = (t', y^{1:i'})$, the single trajectory transition density at time k is

$$\begin{aligned} g^k(Y|X) &= |\tau^{k-1}(X)| \left[(1 - p_S(x^i)) \delta(Y - X) \right. \\ &\quad \left. + p_S(x^i) g(y^{i'}|x^i) \delta\left(\left(t', y^{1:i'-1}\right) - X\right) \right] \\ &\quad + (1 - |\tau^{k-1}(X)|) \delta(Y - X) \end{aligned} \quad (27)$$

That is, if the trajectory has died before time step $k - 1$, the trajectory remains unaltered with probability one. If the trajectory exists at time step $k - 1$, it remains unaltered (target dies) with probability $(1 - p_S(\cdot))$ or the last target state is generated according to the single target transition density with probability $p_S(\cdot)$. Without target births, the number of trajectories remains unchanged so the transition density for the sets of trajectories is [9, Eq. (12.91)]

$$g^k(\{Y_1, \dots, Y_n\} | \{X_1, \dots, X_n\}) = \sum_{\sigma \in \Gamma_n} \prod_{j=1}^n g^k(Y_{\sigma_j} | X_j) \quad (28)$$

where Γ_n is the set that includes all the permutations of $(1, \dots, n)$.

We evaluate $\pi^{k|k-1}(\cdot)$ for $\mathbf{W} \uplus \mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}$, defined in Theorem 5. The multitrajectory PDF for trajectories $\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}$ (no births) is denoted by $\pi_S^{k|k-1}(\cdot)$ and the multitrajectory PDF for \mathbf{W} by $\beta^k(\cdot)$. Using the convolution formula [10, Sec. 4.2.3],

$$\pi^{k|k-1}(\mathbf{W} \uplus \mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}) = \pi_S^{k|k-1}(\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}) \beta^k(\mathbf{W})$$

where

$$\beta^k(\mathbf{W}) = \beta_\tau(\tau^k(\mathbf{W}))$$

as the states of new born trajectories are distributed according to the multiobject PDF $\beta_\tau(\cdot)$. Using (28), we have

$$\pi_S^{k|k-1}(\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}) = \int g^k(\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z} | \mathbf{W}) \pi^{k-1}(\mathbf{W}) \delta \mathbf{W}.$$

Partitioning $\mathbf{W} = \mathbf{D} \uplus \mathbf{A}$, where \mathbf{D} and \mathbf{A} represent dead and alive trajectories at time $k - 1$, we have [10, Sec. 3.5.3]

$$\begin{aligned} &\pi_S^{k|k-1}(\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z}) \\ &= \int g^k(\mathbf{X} \uplus \mathbf{Y} \uplus \mathbf{Z} | \mathbf{D} \uplus \mathbf{A}) \pi^{k-1}(\mathbf{D} \uplus \mathbf{A}) \delta \mathbf{D} \delta \mathbf{A} \\ &= \int g^k(\mathbf{X} \uplus \mathbf{Y} | \mathbf{A}) \pi^{k-1}(\mathbf{Z} \uplus \mathbf{A}) \delta \mathbf{A} \\ &= \int \prod_{j=1}^m g^k(X_j | A_j) \prod_{j=m+1}^{m+n} g^k(Y_{j-m} | A_j) \\ &\quad \times \pi^{k-1}(\mathbf{Z} \uplus \{A_1, \dots, A_{m+n}\}) dA_{m+n}. \end{aligned} \quad (29)$$

where $\mathbf{X} = \{X_1, \dots, X_m\}$ and $\mathbf{Y} = \{Y_1, \dots, Y_n\}$. The proof is finished by substituting (27) into (29).

APPENDIX C

In this appendix, we obtain (21). If new born targets are an IID cluster RFS, the PDF of a trajectory given a hypothesis does not depend on l so we can write

$$p^{k|k}(X_j|l_j, t_j, i_j, \xi_j) = p^{k|k}(X_j|t_j, i_j, \xi_j).$$

Then, (10) becomes

$$\begin{aligned} & \pi^k(\{X_1, \dots, X_n\}) \\ &= \sum_{(t_1, i_1, \xi_1, \dots, t_n, i_n, \xi_n)} \prod_{j=1}^n p^{k|k}(X_j|t_j, i_j, \xi_j) \\ & \times \sum_{l_{1:n}: (l_1, i_1) \neq \dots \neq (l_n, i_n)} w^{k|k}(\{(l_1, t_1, i_1, \xi_1), \dots, (l_n, t_n, i_n, \xi_n)\}) \end{aligned} \quad (30)$$

We can define the ordered weights

$$\begin{aligned} & \hat{w}^{k|k}((t_1, i_1, \xi_1), \dots, (t_n, i_n, \xi_n)) \\ &= \sum_{l_{1:n}: (l_1, t_1) \neq \dots \neq (l_n, t_n)} w^{k|k}(\{(l_1, t_1, i_1, \xi_1), \dots, (l_n, t_n, i_n, \xi_n)\}) \end{aligned}$$

and hypotheses without initial birth component as $\hat{h}_j^{k|k} = (t_j, i_j, \xi_j)$ such that the multitrajectory filtering PDF is written as in (21).

APPENDIX D

Here, we prove Theorem 9. If we have multitrajectory PDF $\pi(\cdot)$ and a transition kernel $f(\cdot|\cdot)$, the multitrajectory PDF $\pi'(\cdot)$ of the transitioned set of trajectories is [10]

$$\pi'(\mathbf{Y}) = \int f(\mathbf{Y}|\mathbf{X}) \pi(\mathbf{X}) \delta \mathbf{X}$$

which is analogous to the prediction step (6). An RFS of targets is an RFS of trajectories whose lengths are one and the start time information has been discarded. Consequently, the same type of relation holds if the transition is for RFS of targets,

$$\pi'(\mathbf{y}) = \int f(\mathbf{y}|\mathbf{X}) \pi(\mathbf{X}) \delta \mathbf{X}.$$

In Theorem 9, the transition is $\mathbf{y} = \tau^k(\mathbf{X})$. Therefore, it is represented by a multitarget Dirac delta $f(\mathbf{y}|\mathbf{X}) = \delta_{\tau^k(\mathbf{X})}(\mathbf{y})$ and we obtain Theorem 9.

APPENDIX E

This appendix shows the relations in Figure 3. The update step is trivial so we focus on the prediction. Theorem 9 can be written as

$$\begin{aligned} & \pi_\tau^k(\{z_1, \dots, z_m\}) \\ &= \sum_{n=m}^{\infty} \frac{1}{n!} \int \delta_{\tau^k(\{X_1, \dots, X_n\})}(\{z_1, \dots, z_m\}) \\ & \times \pi(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= m! \sum_{n=m}^{\infty} \frac{1}{n!} c_m^n \int \left[\prod_{j=1}^m \delta_{\tau^k(X_j)}(\{z_j\}) \right] \\ & \times \left[\prod_{j=m+1}^n \delta_{\tau^k(X_j)}(\emptyset) \right] \pi(\{X_1, \dots, X_n\}) dX_{1:n}. \end{aligned} \quad (31)$$

Prediction without births (targets): The prior multitarget PDF without target births is [9, Sec 13.2.3]

$$\begin{aligned} & \pi_{S,\tau}^{k|k-1}(\{z_1, \dots, z_m\}) \\ &= m! \sum_{n=m}^{\infty} \frac{1}{n!} c_m^n \int \left[\prod_{i=1}^n (1 - p_S(y_i)) \right] \\ & \times \left[\prod_{i=1}^m \frac{p_S(y_i) g(z_i|y_i)}{1 - p_S(y_i)} \right] \pi_\tau^{k-1}(\{y_1, \dots, y_n\}) dy_{1:n} \end{aligned} \quad (32)$$

where $m!c_{n,m}$ indicates the total number of associations from n elements to m elements.

Prediction without births (trajectories): We compute the multitarget PDF $a_\tau^k(\cdot)$ at time k from the predicted multitrajectory PDF without target births $\pi_S^{k|k-1}(\cdot)$, via Theorem 9, and show that is equal to $\pi_{S,\tau}^{k|k-1}(\cdot)$, see Eq. (32). Using (31), we get

$$\begin{aligned} & a_\tau^k(\{z_1, \dots, z_m\}) \\ &= m! \sum_{n=m}^{\infty} \frac{1}{n!} c_m^n \int \left[\prod_{j=1}^m \delta_{\tau^k(X_j)}(\{z_j\}) \right] \\ & \times \left[\prod_{j=m+1}^n \delta_{\tau^k(X_j)}(\emptyset) \right] \pi_S^{k|k-1}(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= m! \sum_{n=m}^{\infty} \frac{1}{n!} c_m^n \int \left[\prod_{j=1}^m (1 - \delta_{\tau^{k-1}(X_j)}(\emptyset)) \right] \\ & \times \left[\prod_{j=1}^n (1 - p_S(X_j^l))^{\lfloor \tau^{k-1}(X_j) \rfloor} \right] \\ & \times \left[\prod_{j=1}^m \frac{g(z_j|X_j^l) p_S(X_j^l)}{1 - p_S(X_j^l)} \right] \pi^{k-1}(\{X_1, \dots, X_n\}) dX_{1:n} \end{aligned}$$

where we have used Theorem 5 and X_j^l denotes the last target state of trajectory X_j . Let p denote the number of targets present at time $k-1$, which satisfies $m \leq p \leq n$. Then, there are $p-m$ targets present at time $k-1$ that are not present at time k . We can write the previous equation as

$$\begin{aligned} & a_\tau^k(\{z_1, \dots, z_m\}) \\ &= m! \sum_{n=m}^{\infty} \sum_{p=m}^n \frac{1}{n!} c_m^n c_{p-m}^{n-m} \int \left[\prod_{j=1}^p (1 - \delta_{\tau^{k-1}(X_j)}(\emptyset)) \right] \\ & \times \left[\prod_{j=p+1}^n \delta_{\tau^{k-1}(X_j)}(\emptyset) \right] \left[\prod_{j=1}^p (1 - p_S(X_j^l)) \right] \\ & \times \left[\prod_{j=1}^m \frac{g(z_j|X_j^l) p_S(X_j^l)}{1 - p_S(X_j^l)} \right] \pi^{k-1}(\{X_1, \dots, X_n\}) dX_{1:n} \\ &= m! \sum_{p=m}^{\infty} \sum_{n=p}^{\infty} \frac{1}{n!} c_p^n c_m^p \int \int \left[\prod_{j=1}^p \delta_{\tau^{k-1}(X_j)}(\{y_j\}) \right] \end{aligned}$$

$$\begin{aligned}
& \times \left[\prod_{j=p+1}^n \delta_{\tau^{k-1}(X_j)}(\odot) \right] \left[\prod_{j=1}^p (1 - p_S(y_j)) \right] \\
& \times \left[\prod_{j=1}^m \frac{g(z_j | y_j) p_S(y_j)}{1 - p_S(y_j)} \right] \pi^{k-1}(\{X_1, \dots, X_n\}) dX_{1:n} dy_{1:p} \\
& = m! \sum_{p=m}^{\infty} \frac{1}{p!} c_m^p \int \left[\prod_{j=1}^p (1 - p_S(y_j)) \right] \\
& \times \left[\prod_{j=1}^m \frac{g(z_j | y_j) p_S(y_j)}{1 - p_S(y_j)} \right] \pi_{\tau}^{k-1}(\{y_1, \dots, y_p\}) dy_{1:p} \\
& = \pi_{S,\tau}^{k|k-1}(\{z_1, \dots, z_m\}).
\end{aligned}$$

Prediction with births (targets): Using the convolution formula [9], the prior at time k is

$$\pi_{\tau}^{k|k-1}(\mathbf{y}) = \sum_{\mathbf{w} \subseteq \mathbf{y}} \pi_{S,\tau}^{k|k-1}(\mathbf{w}) \beta_{\tau}(\mathbf{y} \setminus \mathbf{w}). \quad (33)$$

Prediction with births (trajectories): Using Theorem 9,

$$\begin{aligned}
\pi_{S,\tau}^{k|k-1}(\mathbf{w}) &= \int \delta_{\tau^k(\mathbf{X})}(\mathbf{w}) \pi_S^{k|k-1}(\mathbf{X}) \delta \mathbf{X} \\
\beta_{\tau}(\mathbf{y}) &= \int \delta_{\tau^k(\mathbf{X})}(\mathbf{y}) \beta^k(\mathbf{X}) \delta \mathbf{X}.
\end{aligned}$$

We have that for RFS of trajectories

$$\begin{aligned}
\pi^{k|k-1}(\mathbf{Y}) &= \sum_{\mathbf{W} \subseteq \mathbf{Y}} \pi_S^{k|k-1}(\mathbf{Y} \setminus \mathbf{W}) \beta^k(\mathbf{W}) \\
&= \pi_S^{k|k-1}(\mathbf{Y} \cap \mathbb{S}') \beta^k(\mathbf{Y} \cap \mathbb{B}')
\end{aligned}$$

where \mathbb{S}' and \mathbb{B}' are the disjoint spaces of sets of trajectories that survive at time k and are born at time k , respectively.

By applying the convolution formula to multitarget Dirac deltas, it is met that

$$\delta_{\mathbf{a} \uplus \mathbf{b}}(\mathbf{y}) = \sum_{\mathbf{w} \subseteq \mathbf{y}} \delta_{\mathbf{a}}(\mathbf{w}) \delta_{\mathbf{b}}(\mathbf{y} \setminus \mathbf{w}).$$

The marginal PDF of $\pi^{k|k-1}(\cdot)$ is denoted as $b_{\tau}(\cdot)$. In the following, we show that it is equal to $\pi_{\tau}^{k|k-1}(\cdot)$, see (33). Using Theorem 9,

$$\begin{aligned}
b_{\tau}(\mathbf{y}) &= \int \delta_{\tau^k(\mathbf{X})}(\mathbf{y}) \pi^{k|k-1}(\mathbf{X}) \delta \mathbf{X} \\
&= \int \delta_{\tau^k(\mathbf{X})}(\mathbf{y}) \pi_S^{k|k-1}(\mathbf{X} \cap \mathbb{S}') \beta^k(\mathbf{X} \cap \mathbb{B}') \delta \mathbf{X} \\
&= \int \sum_{\mathbf{w} \subseteq \mathbf{y}} \delta_{\tau^k(\mathbf{X} \cap \mathbb{S}')}(\mathbf{w}) \delta_{\tau^k(\mathbf{X} \cap \mathbb{B}')}(\mathbf{y} \setminus \mathbf{w}) \\
&\quad \times \pi_S^{k|k-1}(\mathbf{X} \cap \mathbb{S}') \beta^k(\mathbf{X} \cap \mathbb{B}') \delta \mathbf{X} \\
&= \sum_{\mathbf{w} \subseteq \mathbf{y}} \int \delta_{\tau^k(\mathbf{X} \cap \mathbb{S}')}(\mathbf{w}) \pi_S^{k|k-1}(\mathbf{X} \cap \mathbb{S}') \\
&\quad \times \delta_{\tau^k(\mathbf{X} \cap \mathbb{B}')}(\mathbf{y} \setminus \mathbf{w}) \beta^k(\mathbf{X} \cap \mathbb{B}') \delta \mathbf{X}.
\end{aligned}$$

Using the properties of the set integrals in joint spaces [10, Sec. 3.5.3], if $\mathbf{X} = \mathbf{X}_1 \uplus \mathbf{X}_2$, then

$$\int f(\mathbf{X}_1) g(\mathbf{X}_2) \delta \mathbf{X} = \int f(\mathbf{X}_1) \delta \mathbf{X}_1 \int g(\mathbf{X}_2) \delta \mathbf{X}_2.$$

Therefore,

$$\begin{aligned}
b_{\tau}(\mathbf{y}) &= \sum_{\mathbf{w} \subseteq \mathbf{y}} \left[\int \delta_{\tau^k(\mathbf{X})}(\mathbf{w}) \pi_S^{k|k-1}(\mathbf{X}) \delta \mathbf{X} \right] \\
&\quad \times \left[\int \delta_{\tau^k(\mathbf{X})}(\mathbf{y} \setminus \mathbf{w}) \beta^k(\mathbf{X}) \delta \mathbf{X} \right] \\
&= \sum_{\mathbf{w} \subseteq \mathbf{y}} \pi_{S,\tau}^{k|k-1}(\mathbf{w}) \beta_{\tau}(\mathbf{y} \setminus \mathbf{w})
\end{aligned}$$

which finishes the proof by comparison with (33).

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